$X^2 - Y^2 + \sqrt{2}\,X - 7\sqrt{2} - 20 = 0$ Ans.

(iii) $\mathbf{5x^2 - 6xy + 5y^2 - 8 = 0}$

Solution:

$5x^2 - 6xy + 5y^2 - 8 = 0$ (1)

We know that for rotation of axes

$x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$ Put in (1)

$5(X\cos\theta - Y\sin\theta)^2 - 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 5(X\sin\theta + Y\cos\theta)^2 - 8 = 0$

$5(X^2\cos^2\theta + Y^2\sin^2\theta - 2XY\sin\theta\cos\theta) - 6(X^2\cos\theta\sin\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta) + 5(X^2\sin^2\theta + Y^2\cos^2\theta + 2XY\sin\theta\cos\theta) - 8 = 0$

$5X^2\cos^2\theta + 5Y^2\sin^2\theta - 10XY\sin\theta\cos\theta - 6X^2\cos\theta\sin\theta - 6XY\cos^2\theta + 6XY\sin^2\theta + 6Y^2\sin\theta\cos\theta + 5X^2\sin^2\theta + 5Y^2\cos^2\theta + 10XY\sin\theta\cos\theta - 8 = 0$

$X^2(5\cos^2\theta - 6\cos\theta\sin\theta + 5\sin^2\theta) + XY(-10\cos\theta\sin\theta - 6\cos^2\theta + 6\sin^2\theta + 10\sin\theta\cos\theta) + Y^2(5\sin^2\theta + 6\sin\theta\cos\theta + 5\cos^2\theta) - 8 = 0$ ..... (2)

To remove XY terms put (2)

$-6\cos^2\theta + 6\sin^2\theta = 0$

$\cos^2\theta = \sin^2\theta \Rightarrow \tan^2\theta = 1$

$\tan\theta = 1$ (θ is taken is 1st Quadrant)

$\theta = 45^\circ$ Put in (2)

$X^2(5(\cos 45^\circ)^2 - 6\cos 45^\circ \sin 45^\circ + 5(\sin 45^\circ)^2) + 0 + Y^2(5(\sin 45^\circ)^2 + 6\sin 45^\circ\cos 45^\circ + 5(\cos 45^\circ)^2) - 8 = 0$

$$X^2\left(5 + \left(\frac{1}{\sqrt{2}}\right)^2 - 6\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + 5\left(\frac{1}{\sqrt{2}}\right)^2\right) + Y^2\left(5 + \left(\frac{1}{\sqrt{2}}\right)^2 + 6\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + 5\left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$X^2\left(\frac{5}{2} - \frac{6}{2} + \frac{5}{2}\right) + Y^2\left(\frac{5}{2} + \frac{6}{2} + \frac{5}{2}\right) - 8 = 0$$

$2X^2 + 8Y^2 - 8 = 0$

$\Rightarrow$ $X^2 + 4Y^2 - 4 = 0$ Ans

EXERCISE 6.9

Q.1: By rotation of axes, eliminates the xy-term in each of the following equations. Identify the conic & find its elements.

(i) $\mathbf{4x^2 - 4xy + y^2 - 6 = 0}$

Solution:

$4x^2 - 4xy + y^2 - 6 = 0$ (1)

Here $a = 4$ $b = 1$ $2h = -4$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{-4}{4-1} = \frac{-4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{-4}{3}$$

$$6\tan\theta = -4 + 4\tan^2\theta$$

$$4\tan^2\theta - 6\tan\theta - 4 = 0 \quad \Rightarrow \quad 2\tan^2\theta - 3\tan\theta - 2 = 0$$

$$2\tan^2\theta - 4\tan\theta + \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta - 2) + 1(\tan\theta - 2) = 0$$

$$(\tan\theta - 2)(2\tan\theta + 1) = 0$$

$$\tan\theta = 2, \ \tan\theta = \frac{-1}{2} \text{ (is not admissible)}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+4} = \sqrt{5}$$

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{5}}$$

$$= \sqrt{\frac{5-1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

Since equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right] \text{ putting values of } \sin\theta \ \& \ \cos\theta \text{ in above equations}$$

$$\left.\begin{aligned} x &= X\frac{1}{\sqrt{5}} - Y\frac{2}{\sqrt{5}} = \frac{X-2Y}{\sqrt{5}} \\ x &= X\frac{2}{\sqrt{5}} - Y\frac{1}{\sqrt{5}} = \frac{2X+Y}{\sqrt{5}} \end{aligned}\right] \quad (2)$$

Using (2) in (1)

$$4\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right) + \left(\frac{2X+Y}{\sqrt{5}}\right)^2 - 6 = 0$$

$$4\left(\frac{X^2+4Y^2-4XY}{5}\right) - 4\left(\frac{2X^2+XY-4XY-2Y^2}{5}\right) + \left(\frac{4X^2+Y^2+4XY}{5}\right) - 6 = 0$$

$$4X^2 + 16Y^2 - 16XY - 8X^2 - 4XY + 16XY + 8Y^2 + 4X^2 + Y^2 + 4XY - 30 = 0$$

$$25Y^2 - 30 = 0$$

$$25Y^2 = 30$$

$$Y^2 = \frac{30}{25} \quad \Rightarrow \quad Y^2 = \frac{6}{5}$$

=> $Y = \pm\sqrt{\frac{6}{5}} \rightarrow$ (3)

From (2) we have $X - 2Y = \sqrt{5}\,x$ ..... (4) , $2X + Y = \sqrt{5}\,y$ ..... (5)

Multiply eq. (4) by 2 & subtract from 5

$$\begin{array}{rcl} 2X + Y &=& \sqrt{5}\,y \\ -2X \mp 4Y &=& -2\sqrt{5}\,x \\ \hline 5Y &=& \sqrt{5}\,y - 2\sqrt{5}\,x \end{array}$$

$$5Y = \sqrt{5}\,(y - 2x)$$

$$Y = -\frac{\sqrt{5}}{5}(2x - y) \rightarrow \quad (6)$$

Using (3) in 6

$$\pm\sqrt{\frac{6}{5}} = -\frac{\sqrt{5}}{5}(2x - y)$$

$$\pm\frac{\sqrt{6}}{\sqrt{5}} = -\frac{1}{\sqrt{5}}(2x - y)$$

$$2x - y \pm \sqrt{6} = 0$$

$2x - y + \sqrt{6} = 0$, $2x - y - \sqrt{6} = 0$ Ans.

(ii) $\mathbf{x^2 - 2xy + y^2 - 8x - 8y = 0}$ **(1)**

Solution:

Here $a = 1$, $2h = -2$, $b = 1$

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-2}{1 - 1} = \frac{-2}{0} = \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty = 90^o$$

$$\theta = 45^o$$

Equations of transformation are

$x = X\cos\theta - Y\sin\theta$ => $x = X\cos 45^o - Y\sin 45^o$

$$\left.\begin{array}{l} x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y = X\sin\theta + Y\cos\theta \;\Rightarrow\; y = X\sin 45^o + Y\cos 45^o \\ y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{array}\right\} \rightarrow \quad (2)$$

Using (2) in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 - 8\left(\frac{X-Y}{\sqrt{2}}\right) - 8\left(\frac{X+Y}{\sqrt{2}}\right) = 0$$

$$\frac{X^2+Y^2-2XY}{2} - \frac{2}{2}(X^2-Y^2) + \frac{X^2+Y^2+2XY}{2} - \frac{8}{\sqrt{2}}(X-Y) - \frac{8}{\sqrt{2}}(X+Y) = 0$$

$$X^2+Y^2-2XY-2(X^2-Y^2)+X^2+Y^2+2XY-8\sqrt{2}(X-Y)-8\sqrt{2}(X+Y)=0$$

$$X^2+Y^2-2XY-2X^2+2Y^2+X^2+Y^2+2XY-8\sqrt{2}X+8\sqrt{2}Y-8\sqrt{2}X-8\sqrt{2}Y=0$$

$$4Y^2 - 16\sqrt{2}X = 0$$

$$Y^2 - 4\sqrt{2}X = 0$$ which represents a Parabola

To solve (2) for values of X & Y

$$x = \frac{X-Y}{\sqrt{2}} \Rightarrow \sqrt{2}x = X-Y \rightarrow (3)$$

$$y = \frac{X+Y}{\sqrt{2}} \Rightarrow \sqrt{2}y = X+Y \rightarrow (4)$$

Adding (3) & (4)

$$\sqrt{2}(x+y) = 2X$$

$$X = \frac{\sqrt{2}}{2}(x+y) \Rightarrow X = \frac{1}{\sqrt{2}}(x+y)$$

Subtracting (3) & (4)

$$\sqrt{2}x = X-Y$$

$$\underline{-\sqrt{2}y = -X \pm Y}$$

$$\sqrt{2}(x-y) = -2Y$$

$$Y = \frac{\sqrt{2}}{2}(x-y)$$

$$Y = \frac{-1}{\sqrt{2}}(x-y)$$

For vertex of Parabola, put X = 0 , Y = 0

$$0 = \frac{1}{\sqrt{2}}(x+y) \quad , \quad 0 = \frac{-1}{\sqrt{2}}(x-y)$$

$$\Rightarrow x+y=0 \qquad \Rightarrow = x-y=0$$

$$x+y=0 \qquad x+y=0$$

Adding $\underline{x-y=0}$ Subtract $\underline{-x \mp y = 0}$

$$2x = 0 \Rightarrow \boxed{x=0} \qquad 2y = 0 \Rightarrow \boxed{y=0}$$

$$\boxed{\text{Vertex} = (0, 0)}$$

Axis of Parabola is $Y = 0$

$$\frac{-1}{\sqrt{2}}(x - y) = 0$$

$$x - y = 0 \quad \Rightarrow \quad \boxed{x = y}$$

Since $Y^2 = 4\sqrt{2}\,X$

As standard form is

$$y^2 = 4aX$$
$$4a = 4\sqrt{2}$$
$$a = \sqrt{2}$$

Focus of Parabola is (0, 0)

$$(X, Y) = (\sqrt{2}, 0)$$

$$\left[\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)\right] = (\sqrt{2}, 0)$$

$$\frac{1}{\sqrt{2}}(x + y) = \sqrt{2} \quad , \quad \frac{1}{\sqrt{2}}(x - y) = 0$$

$$x + y = 2 \qquad x - y = 0$$

Solving above equations

$$x + y = 2$$
$$\underline{x - y = 2}$$
$$2x = 2 \qquad \boxed{x = 1} \qquad 1 - y = 0$$
$$\boxed{y = 1}$$

$\boxed{\text{Focus } (1, 1)}$ Ans

(iii) $\mathbf{x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0}$

Solution:

$$x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0 \quad ..... \quad (1)$$

$$a = 1 \quad \& \quad b = 1 \quad \Rightarrow \quad a = b \;,\; 2h = 2$$

$$\tan 2\theta = \frac{2h}{a - b}$$

$$\tan 2\theta = \frac{2}{0} = \infty$$

$$2\theta = 90^\circ \quad \Rightarrow \quad \theta = 45^\circ$$

Now equations of transformation are

$$x = X\cos\theta - Y\sin\theta = X\cos 45^\circ - Y\sin 45^\circ$$

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}}$$

$$y = X \sin\theta + Y \cos\theta$$

$$y = X \sin 45^\circ + Y \cos 45^\circ$$

$$y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}}$$

$$x = \frac{X - Y}{\sqrt{2}} \quad , \quad y = \frac{X + Y}{\sqrt{2}} \quad \Big] \quad \rightarrow \quad (2)$$

Using (2) in (1)

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{X - Y}{\sqrt{2}}\right) - 2\sqrt{2}\left(\frac{X + Y}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{X^2 + Y^2 - 2XY}{2} + \frac{2}{2}(X^2 - Y^2) + \frac{X^2 + Y^2 + 2XY}{2} + 2(X - Y) - 2(X + Y) + 2 = 0$$

$$X^2 + Y^2 - 2XY + 2X^2 - 2Y^2 + X^2 + Y^2 + 2XY + 4X - 4Y - 4X - 4Y + 4 = 0$$

$$4X^2 - 8Y + 4 = 0 \quad \Rightarrow \quad X^2 - 2Y + 1 = 0$$

$$X^2 = 2Y - 1 \quad \Rightarrow \quad X^2 = 2(Y - 1) \rightarrow \quad (3)$$

Which represents a Parabola

Now, solve (2) for X & Y

Adding

$$\begin{aligned} X + Y &= \sqrt{2}\, y \\ X - Y &= \sqrt{2}\, x \\ \hline 2X &= \sqrt{2}\,(x + y) \end{aligned}$$

Subtracting

$$\begin{aligned} X - Y &= \sqrt{2}\, y \\ -X \pm Y &= -\sqrt{2}\, y \\ \hline -2Y &= \sqrt{2}\,(x - y) \end{aligned}$$

$$Y = \frac{-1}{\sqrt{2}}(x - y)$$

$$X = \frac{\sqrt{2}}{2}(x + y) \quad \Rightarrow \quad X = \frac{1}{\sqrt{2}}(x + y)$$

For vertex of Parabola

Put $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x + y) = 0 \qquad -\frac{1}{\sqrt{2}}(x - y) = 0$$

$$x + y = 0 \qquad x - y = 0$$

$$\begin{aligned} x + y &= 0 \\ x - y &= 0 \\ \hline 2x &= 0 \end{aligned} \quad \Rightarrow \boxed{x = 0} \qquad \begin{aligned} x + y &= 0 \\ -x \mp y &= -0 \\ \hline 2y &= 0 \end{aligned} \quad \Rightarrow \boxed{y = 0}$$

$$\boxed{\text{Vertex} = (0, 0)}$$

Axis of Parabola

$$X = 0$$

$$\frac{1}{\sqrt{2}}(x + y) = 0 \quad \Rightarrow \quad \boxed{x + y = 0}$$

Focus $(X, Y) = (0, a)$

$$\therefore X^2 = 2\left(Y - \frac{1}{2}\right)$$

$$x^2 = 4ay$$

$$\Rightarrow 4a = 2 \quad \boxed{a = \frac{1}{2}}$$

$$\left(\frac{1}{\sqrt{2}}(x + y), \frac{-1}{\sqrt{2}}(x - y)\right) = \left(0, -\frac{1}{2}\right)$$

$$\frac{1}{\sqrt{2}}(x + y) = 0 \qquad \frac{-1}{\sqrt{2}}(x - y) = \frac{1}{2}$$

$$(x + y) = 0 \qquad (x - y) = -\sqrt{2}$$

$$\begin{aligned} x + y &= 0 \\ x - y &= -\sqrt{2} \\ \hline 2x &= -\sqrt{2} \end{aligned} \quad \Rightarrow \boxed{x = \frac{-1}{\sqrt{2}}}$$

$$x - y = -\sqrt{2}$$

$$\frac{-1}{\sqrt{2}} - y = -\sqrt{2}$$

$$\frac{-1 - \sqrt{2}\,y}{\sqrt{2}} = -2$$

$$-1 - \sqrt{2} = -2$$

$$-1 + 2 = \sqrt{2}\,y$$

$$1 = \sqrt{2}\,y \quad \Rightarrow \quad y = \frac{1}{\sqrt{2}}$$

Focus $= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ Ans.

(iv) $\mathbf{x^2 + xy + y^2 - 4 = 0}$

Solution:

Here $a = 1$, $b = 1$, $2h = 1$

If $a = b$ $\theta = 45^o$ (always)

New equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X\cos 45^o - Y\sin 45^o = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \\ y &= X\sin\theta + Y\cos\theta = Y\sin 45^o + Y\cos 45^o = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} \end{aligned}\right] \qquad (2)$$

Putting in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + \left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 - 4 = 0$$

$$\frac{X^2+Y^2-2XY}{2} + \frac{X^2-Y^2}{2} + \frac{X^2+Y^2-2XY}{2} - 4 = 0$$

$$X^2 + Y^2 - 2XY + X^2 - Y^2 + X^2 + Y^2 + 2XY - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{3X^2}{8} + \frac{Y^2}{8} = 1 \quad \Rightarrow \quad \frac{X^2}{\frac{8}{3}} + \frac{Y^2}{8} = 3 \quad \text{(which represents an ellipse)}$$

Solve (2) for X & Y

$$\begin{aligned} \sqrt{2}\,x &= X - Y \\ \sqrt{2}\,y &= X + Y \\ \hline \sqrt{2}\,(x+y) &= 2X \\ X &= \frac{1}{\sqrt{2}}(x+y) \end{aligned} \qquad \begin{aligned} \sqrt{2}\,x &= X - Y \\ -\sqrt{2}\,y &= -X \pm Y \\ \hline \sqrt{2}\,(x-y) &= -2Y \\ Y &= -\frac{1}{\sqrt{2}}(x-y) \end{aligned}$$

Adding

For center put $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x+y) = 0 \qquad \frac{-1}{\sqrt{2}}(x-y) = 0$$

$$(x+y) = 0 \quad , \quad x - y = 0$$

$$\begin{aligned} x + y &= 0 \\ x - y &= 0 \\ \hline 2x &= 0 \end{aligned} \Rightarrow \boxed{x = 0} \qquad \begin{aligned} x + y &= 0 \\ -x \mp y &= -0 \\ \hline 2y &= 0 \end{aligned} \Rightarrow \boxed{y = 0}$$

Adding

Required center (0, 0)

From (3) we have $a^2 = 8$ (length of major axis)

$b^2 = \frac{8}{3}$ (Length of minor axis)

$$\therefore \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{8 - \frac{8}{3}}{8} = \frac{24-8}{3} \times \frac{1}{8} = \frac{16}{3} \times \frac{1}{8} = \frac{2}{3}$$

$$\Rightarrow \quad e = \sqrt{\frac{2}{3}}$$

Coordinates of foci

$= (0, \pm ae)$

$$(X, Y) = (0, \pm 2\sqrt{2}\sqrt{\frac{2}{3}})$$

$$\left(\frac{1}{\sqrt{2}}(x+y), \frac{-1}{\sqrt{2}}(x-y)\right) = \left(0, \pm\frac{4}{\sqrt{3}}\right)$$

$$\frac{1}{\sqrt{2}}(x+y) = 0 \quad , \quad \frac{-1}{\sqrt{2}}(x-y) = \pm\frac{4}{\sqrt{3}}$$

$$\Rightarrow \quad x + y = 0 \qquad x - y = \pm\frac{4\sqrt{2}}{\sqrt{3}}$$

Now $x + y = 0$

Adding $x - y = \pm\frac{4\sqrt{2}}{\sqrt{3}}$

$$2x = \pm\frac{4\sqrt{2}}{\sqrt{3}} \quad \Rightarrow \quad \boxed{x = \pm\frac{2\sqrt{2}}{\sqrt{3}}} \quad \boxed{y = \pm\frac{2\sqrt{2}}{\sqrt{3}}}$$

Coordinates of foci are $\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, \pm\frac{2\sqrt{2}}{\sqrt{3}}\right)$ Ans.

Vertices of ellipse are $(0, \pm a)$

$(X, Y) = (0, \pm 2\sqrt{2})$

$$\frac{1}{\sqrt{2}}(x-y) = 0 \qquad \frac{-1}{\sqrt{2}}(x+y) = \pm 2\sqrt{2}$$

$$\Rightarrow \quad x - y = 0 \qquad x + y = \pm 4$$

$$x - y = 0$$
$$\underline{x + y = \pm 4}$$
$$2x = \pm 4 \quad \Rightarrow \quad \boxed{x = \pm 2}$$
$$y = \pm 2$$

Required vertices are $(\pm 2, \pm 2)$

For equation of major axis Put $Y = 0$

$$\frac{-1}{\sqrt{2}}(x + y) = 0 \quad \Rightarrow \quad x + y = 0$$

For equation of minor axis Put $X = 0$

$$\frac{1}{\sqrt{2}}(x - y) = 0 \quad \Rightarrow \quad x - y = 0$$

(v) $\mathbf{7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0}$

Solution:

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0 \quad .... \quad (1)$$

$$a = 7 \quad , \quad 2h = -6\sqrt{3} \quad , \quad b = 13$$

$$\tan 2\theta = \frac{2h}{a - b}$$

$$\tan 2\theta = \frac{-6\sqrt{3}}{7 - 13} = \frac{-6\sqrt{3}}{-6}$$

$$\tan 2\theta = \sqrt{3} \quad \Rightarrow \quad 2\theta = \tan^{-1}(\sqrt{3}) = 60^o$$

$$\theta = 30^o$$

Now equations of transformation are

$$\left.\begin{aligned} x &= X\cos 30^o - Y\sin 30^o = X\frac{\sqrt{3}}{2} - Y\frac{1}{2} = \frac{\sqrt{3}X - Y}{2} \\ y &= X\sin 30^o + Y\cos 30^o = X\frac{1}{2} + Y\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}Y}{2} \end{aligned}\right] \quad ..... (2)$$

Putting these values in (1)

$$7\left(\frac{\sqrt{3}X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + \sqrt{3}Y}{2}\right) + 13\left(\frac{X + \sqrt{3}Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 + Y^2 - 2\sqrt{3}XY}{4}\right) - \frac{-3\sqrt{3}}{2}(\sqrt{3}X^2 + 3XY - XY - \sqrt{3}y^2) + \frac{13}{4}(X^2 + 3Y^2 + 2\sqrt{3}XY) - 16 = 0$$

$$21X^2 + 7Y^2 - 14\sqrt{3}XY - 6\sqrt{3}(\sqrt{3}X^2 + 2XY - \sqrt{3}Y^2) + 13X^2 + 39Y^2 + 26\sqrt{3}XY - 64 = 0$$

$$21X^2 + 7Y^2 - 14\sqrt{3}\,XY - 18X^2 - 12\sqrt{3}\,XY + 18Y^2 + 13X^2 + 39Y^2 + 26\sqrt{3}\,XY - 64 = 0$$

$$16X^2 + 64Y^2 - 64 = 0$$

$$16X^2 + 64Y^2 = 64$$

$$\frac{X^2}{4} + \frac{Y^2}{1} = 1 \quad \rightarrow \quad (3) \quad \text{which represents on ellipse}$$

Now for values of X & Y solving (2)

$$\begin{aligned} \sqrt{3}\,X - Y &= 2x \\ -\sqrt{3}\,X \pm 3Y &= -2\sqrt{3}\,y \\ \hline -4Y &= 2x - 2\sqrt{3}\,y \\ -4Y &= 2(x - \sqrt{3}\,y) \end{aligned}$$

$$Y = \frac{-1}{2}(x - \sqrt{3}\,y) \;...\; (4)$$

$$\begin{aligned} 3X - \sqrt{3}\,Y &= 2\sqrt{3}\,x \\ X + \sqrt{3}\,Y &= 2y \\ \hline 4X &= 2\sqrt{3}\,x + 2y \end{aligned}$$

$$X = \frac{1}{2}(\sqrt{3}\,x + y) \;...\; (5)$$

From (3), we have

$$a^2 = 4, \qquad b^2 = 1 \qquad \therefore \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4-1}{4} = \frac{3}{4}$$

$$\Rightarrow \quad e = \frac{\sqrt{3}}{2}$$

Coordinates of foci $= (\pm ae, 0)$

$$(X, Y) = (\pm\sqrt{3}, 0)$$

$$\left[\frac{1}{2}(\sqrt{3}\,x + y), \frac{-1}{2}(x - \sqrt{3}\,y)\right] = (\pm\sqrt{3}, 0)$$

$$\frac{1}{2}(\sqrt{3}\,x + y) = \pm\sqrt{3} \qquad \frac{-1}{2}(x - \sqrt{3}\,y) = 0$$

$$\sqrt{3}\,x + y = \pm 2\sqrt{3} \quad (i) \qquad x - \sqrt{3}\,y) = 0 \quad (ii)$$

Multiplying (ii) by $\sqrt{3}$ & subtracting from (i)

$$\begin{aligned} \sqrt{3}\,x + y &= 2\sqrt{3} \\ -\sqrt{3}\,x \mp 3y &= \underline{0} \\ \hline 4y &= 2\sqrt{3} \\ y &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \sqrt{3}\,x + y &= -2\sqrt{3} \\ -\sqrt{3}\,x \mp 3y &= \underline{0} \\ \hline 4y &= -2\sqrt{3} \\ y &= \frac{-1}{2}\sqrt{3} \end{aligned}$$

$$x - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 0 \qquad \therefore \qquad x - \sqrt{3}\left(-\frac{1}{2}\sqrt{3}\right) = 0$$

$$x - \frac{3}{2} = 0 \qquad x + \frac{3}{2} = 0$$

$$x = \frac{3}{2} \qquad x = \frac{-3}{2}$$

Coordinates of foci

$$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \qquad \left(\frac{-3}{2}, \frac{-\sqrt{3}}{2}\right)$$

Vertices

$$(\pm a, 0)$$

$$(X, Y) = (\pm 2, 0)$$

$$\frac{1}{2}(\sqrt{3}x + y) = \pm 2 \quad , \quad \frac{1}{2}(\sqrt{3}y - x) = 0$$

$$\sqrt{3}x + y = \pm 4 \qquad \sqrt{3}y - x) = 0$$

Solving above equations

$$\sqrt{3}x + y = 4 \qquad \text{(iii)}$$

$$\sqrt{3}x + y = -4 \qquad \text{(iv)}$$

$$-x + \sqrt{3}y = 0 \qquad -x + \sqrt{3}y = 0 \qquad \text{(v)}$$

Multiply equation (v) by $\sqrt{3}$ & adding in (iii) and (iv)

$$\begin{array}{rcl} \sqrt{3}x + y &=& 4 \\ -\sqrt{3}x + 3y &=& 0 \\ \hline 4y &=& 4 \end{array} \qquad \begin{array}{rcl} \sqrt{3}x + y &=& -4 \\ -\sqrt{3}x + 3y &=& 0 \\ \hline 4y &=& -4 \end{array}$$

$$4y = 4 \qquad\qquad 4y = -4$$

$$\boxed{y = 1} \qquad\qquad y = -1$$

$$\therefore \quad -x + \sqrt{3}(1) = 0 \qquad\qquad \therefore \quad -x + \sqrt{3}(-1) = 0$$

$$\boxed{x = \sqrt{3}} \qquad\qquad x = -\sqrt{3}$$

Required vertex $(\sqrt{3}, 1)$ & $(-\sqrt{3}, -1)$

For equation of major axes, for equation of minor axes

Put $Y = 0$	Put $X = 0$
$\frac{1}{2}(\sqrt{3}y - x) = 0$	$\frac{1}{2}(\sqrt{3}x + y) = 0$
$\sqrt{3}y - x = 0$	$\sqrt{3}x + y = 0$

(vi) $\mathbf{4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0}$ **(1)**

Solution:

$$a = 4 \;,\; b = 7 \;,\; 2h = -4$$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$= \frac{-4}{4-7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3} \quad \Rightarrow \quad 6\tan\theta = 4 - 4\tan^2\theta$$

$$\Rightarrow \quad 4\tan^2\theta + 6\tan\theta - 4 = 0$$

$$2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$2\tan^2\theta + 4\tan\theta - \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta + 2) - 1(\tan\theta + 2) = 0$$

$$(\tan\theta + 2)(2\tan\theta - 1) = 0$$

$$\tan\theta + 2 = 0, \; 2\tan\theta - 1 = 0$$

$$\tan\theta = -2$$ (neglect it, because θ is taken from 1st Quadrant)

And also

$$2\tan\theta - 1 = 0$$

$$2\tan\theta = 1 \quad \Rightarrow \quad \tan\theta = \frac{1}{2}$$

$$\cot\theta = 2 \quad \Rightarrow \quad \operatorname{cosec}\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+4} = \sqrt{5}$$

$$\sin\theta = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad \cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\frac{1}{5}} = \sqrt{\frac{5-1}{5}} = \frac{2}{\sqrt{5}}$$

Now equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X\left(\frac{2}{\sqrt{5}}\right) - Y\left(\frac{1}{\sqrt{5}}\right) = \frac{2X-Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\cos\theta = X\frac{1}{\sqrt{5}} + Y\frac{2}{\sqrt{5}} = \frac{X+2Y}{\sqrt{5}} \end{aligned}\right] \text{..... (2)}$$

Now putting values in (1)

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2 + 12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$4\left(\frac{4X^2+Y^2-4XY}{\sqrt{5}}\right) - \frac{4}{5}(2X^2+4XY-XY-2Y^2) + \frac{7}{15}(X^2+4Y^2+4XY) + \frac{12}{\sqrt{5}}$$

$(2X - Y) + \frac{6}{\sqrt{5}}\ (X + 2Y) - 9 = 0$

$4(4X^2 - 4XY + Y^2) - 4\ (2X^2 + 3XY - 2Y^2) + 7\ (X^2 + 4Y^2 + 4XY) + 12\sqrt{5}\ (2X - Y)$
$+ 6\sqrt{5}\ (X + 2Y) - 45 = 0$

$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28Y^2 + 28XY + 24\sqrt{5}\ X - 12\sqrt{5}\ Y$
$+ 6\sqrt{5}\ X + 12\sqrt{5}\ Y - 45 = 0$

$15X^2 + 40Y^2 + 30\sqrt{5}\ X - 45 = 0$

=> $3X^2 + 8Y^2 + 6\sqrt{5}\ X - 9 = 0$

$3X^2 + 6\sqrt{5}\ X + 8Y^2 = 9$

$3(X^2 + 2\sqrt{5}\ X + (\sqrt{5})^2 + 8Y^2 = 9 + 3\ (\sqrt{5})^2$

=> $3(X + \sqrt{5})^2 + 8Y^2 = 24$

$\frac{(X + \sqrt{5})^2}{8} + \frac{Y^2}{3} = 1$ ..... (3) which represents on ellipse

Now, for values of X & Y solving (2)

$2X - Y = \sqrt{5}\ x$ $\quad$ $2X - Y = \sqrt{5}\ x$

$X + 2Y = \sqrt{5}\ y$ $\quad$ $X + 2Y = \sqrt{5}\ y$

Multiply 1st Equation by 2 & multiply equation (2) by 2 and adding

$4X - 2y = 2\sqrt{5}\ x$ $\quad$ $2X - Y = \sqrt{5}\ x$

$X + 2y = \sqrt{5}\ y$ $\quad$ $-2X \pm 4Y = -2\sqrt{5}\ y$

$5X = \sqrt{5}\ (2x + y)$ $\quad$ $-5Y = \sqrt{5}\ (x - 2y)$

=> $X = \frac{1}{\sqrt{5}}\ (2x + y)$ ...(4) $\quad$ $Y = \frac{-1}{\sqrt{5}}(x - 2y) \rightarrow (5)$

From (3) $\quad a^2 = 8 \quad \& \quad b^2 = 3$

$e^2 = \frac{a^2 - b^2}{a^2} = \frac{8 - 3}{8} = \frac{5}{8}$ => $e = \sqrt{\frac{5}{8}}$

Coordinates of foci $= (\pm\ ae, 0)$

$(X + \sqrt{5}, Y) = (\pm\sqrt{5}, 0)$

=> $\frac{1}{\sqrt{5}}\ (2x + y) + \sqrt{5} = \pm\sqrt{5}$

$2x + y = \pm 5 - 5$

$2x + y = 0$, $2x + y = -10$

$x + 2y = 0$, $x - 2y = 0$

Multiplying 1st equation by 2

$$\begin{array}{rcl} 4x + 2y &=& 0 \\ -\ x + 2y &=& 0 \\ \hline 3x &=& 0 \end{array} \Rightarrow \boxed{x = 0} \therefore \boxed{y = 0}$$

$$\begin{array}{rcl} 2x + y &=& -10 \\ -\ 2x \mp 4y &=& 0 \\ \hline 5y &=& -10 \end{array}$$

$$\boxed{y = -2} \therefore \boxed{x = -4}$$

Coordinates of foci are

$(0, 0)$ & $(-4, -2)$

Vertices

$(X + \sqrt{5}\ , Y) = (\pm\sqrt{8}, 0)$

$X + \sqrt{5} = \pm\sqrt{8} \qquad Y = 0$

$\frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = \sqrt{8}$, $\frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = -\sqrt{8}$ and $\frac{-1}{\sqrt{5}}(x - 2y) = 0$

$2x + y + 5 = \sqrt{40}$, $2x + y + 5 = -40$ & $x - 2y = 0$ ..... (2)

$\Rightarrow \quad 2x + y = \sqrt{40} - 5$, $2x + y = -40 - 5$

$$\begin{array}{rcl} -\ 2x \mp 4y &=& 0 \\ \hline 5y &=& \sqrt{40} - 5 \\ y &=& \frac{\sqrt{40} - 5}{5} \end{array} \quad \therefore \quad x - 2\left(\frac{\sqrt{40} - 5}{5}\right) = 0$$

$$x = 2\left(\frac{\sqrt{40} - 5}{5}\right)$$

Vertex is $\left(\frac{2\sqrt{40}}{5} - 1\ , \frac{\sqrt{40}}{5} - 1\right)$

(vii) $xy - 4x - 2y = 0$

Solution:

$xy - 4x - 2y = 0 \qquad (1)$

$a = 0$, $b = 0$, $2h = 1$

Since $a = b$ therefore $\theta = 45^o$ (always)

Now equations of transformations are

$$\left.\begin{array}{l} x = X\cos\theta - Y\sin\theta = X\cos 45^o - Y\sin 45^o = \frac{X - Y}{\sqrt{2}} \\ y = X\sin\theta + Y\cos\theta = X\sin 45^o + Y\cos 45^o = \frac{X + Y}{\sqrt{2}} \end{array}\right] \text{..... (2)}$$

Putting values in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) - 4\left(\frac{X-Y}{\sqrt{2}}\right) - 2\left(\frac{X+Y}{\sqrt{2}}\right) = 0$$

$$\frac{X^2-Y^2}{2} - \frac{4}{\sqrt{2}}(X-Y) - \frac{2}{\sqrt{2}}(X+Y) = 0$$

$$X^2 - Y^2 - 4\sqrt{2}\,X + 4\sqrt{2}\,Y - 2\sqrt{2}\,X - 2\sqrt{2}\,Y = 0$$

$$X^2 - 6\sqrt{2}\,X - Y^2 + 2\sqrt{2}\,Y = 0$$

$$(X^2 - 6\sqrt{2}\,X + 18) - (Y^2 - 2\sqrt{2}\,Y + 2) = 18 - 2$$

$$(X^2 - 3\sqrt{2})^2 - (Y - \sqrt{2})^2 = 16$$

$$\frac{(X-3\sqrt{2})^2}{16} - \frac{(Y-\sqrt{2})^2}{16} = 1 \quad \text{.... (3) which represents hyperbola}$$

Now for the values of X & Y, solving (2)

Adding

$$\begin{aligned} X - Y &= \sqrt{2}\,x \\ X + Y &= \sqrt{2}\,y \\ \hline 2X &= \sqrt{2}\,(x+y) \\ X &= \frac{1}{\sqrt{2}}(x+y) \end{aligned}$$

Subtracting

$$\begin{aligned} X - Y &= \sqrt{2}\,x \\ -X \pm Y &= -\sqrt{2}\,y \\ \hline -2Y &= \sqrt{2}\,(x-y) \\ Y &= \frac{-1}{\sqrt{2}}(x-y) \end{aligned}$$

From (3) we have $a^2 = 16$, $b^2 = 16$

$$e^2 = \frac{a^2+b^2}{a^2} = \frac{16+16}{16} = \frac{32}{16} = 2$$

=> $e = \sqrt{2}$

foci $(\pm ae = 0)$

$(X - 3\sqrt{2}, Y - \sqrt{2}) = (\pm 4\sqrt{2} = 0)$

$X - 3\sqrt{2} = \pm 4\sqrt{2}$ $\quad$ $Y - \sqrt{2} = 0$

$X = 3\sqrt{2} \pm 4\sqrt{2}$ $\quad$ $Y = \sqrt{2}$

$\frac{1}{\sqrt{2}}(x+y) = 3\sqrt{2} \pm 4\sqrt{2}$ => $x + y = 6 \pm 8$ => $x + y = 14$, $x + y = -2$

$$\begin{aligned} x - y &= -2 \;, & x - y &= -2 \\ 2x &= 12 & 2x &= -4 \\ \boxed{x = 6} & & \boxed{x = -2} & \end{aligned}$$

=> $x = 6$ & $x = -2$

$6 + y = 14$, $-2 + y = -2$

$\boxed{y = 8}$ $\quad$ $\boxed{y = 0}$ $\quad$ Required foci (6, 8) & (– 2, 0)

Vertices $= (\pm a, 0)$

$(X - 3\sqrt{2}, \; Y - \sqrt{2}) = (\pm 4, 0)$

$X - 3\sqrt{2} = \pm 4 \qquad Y - \sqrt{2} = 0$

$X = 3\sqrt{2} \pm 4 \qquad \frac{-1}{\sqrt{2}}(x - y) = \sqrt{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}(x + y) = 3\sqrt{2} \pm 4 \quad , \quad x - y = -2 \quad \Rightarrow x + y = 6 \pm 4\sqrt{2}$

$$\begin{aligned} x + y &= 6 + 4\sqrt{2} \\ x - y &= -2 \\ \hline 2x &= 4 + 4\sqrt{2} \end{aligned} \qquad \& \qquad \begin{aligned} x + y &= 6 - 4\sqrt{2} \\ x - y &= -2 \\ \hline 2x &= 4 - 4\sqrt{2} \end{aligned}$$

$\Rightarrow \quad x = 2 + 2\sqrt{2} \qquad\qquad x = 2 - 2\sqrt{2}$

$\therefore \quad y = x + 2 \qquad\qquad \therefore \quad y = x + 2$

$= 2 + 2\sqrt{2} + 2 \qquad\qquad = 2 - 2\sqrt{2} + 2$

$y = 4 + 2\sqrt{2} \qquad\qquad y = 4 - 2\sqrt{2}$

Required vertices $(2 + 2\sqrt{2}, 4 + 2\sqrt{2})$ & $(2 - 2\sqrt{2}, 4 - 2\sqrt{2})$

Next, equation of focal axis

$Y - \sqrt{2} = 0$

$Y = \sqrt{2}$

$\frac{-1}{\sqrt{2}}(x - y) = \sqrt{2}$

$x - y = -2$

Equation of conjugate axis $\quad X - 3\sqrt{2} = 0$

$\Rightarrow \quad X = 3\sqrt{2}$

$\frac{1}{\sqrt{2}}(x + y) = 3\sqrt{2}$

$x + y = 6$

(viii) $\mathbf{x^2 + 4xy - 2y^2 - 6 = 0 \qquad (1)}$

Solution:

$a = 1 \;, \quad 2h = 4 \qquad b = -2$

$\tan 2\theta = \frac{2h}{a - b} = \frac{4}{3}$

$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{3}$

$$2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3 \pm \sqrt{25}}{4} = \frac{1}{2}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\cos\theta = \frac{2}{\sqrt{5}} \quad \sin\theta = \frac{1}{\sqrt{5}}$$

Equations of transformation are

$$x = X\cos\theta - Y\sin\theta \Rightarrow \frac{2X-Y}{\sqrt{5}} \quad ..... (2)$$

$$y = X\sin\theta + Y\cos\theta \Rightarrow \frac{X+2Y}{\sqrt{5}} \quad .....(3)$$

Put in (1) we have

$$\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) - 2\left(\frac{X+2Y}{\sqrt{5}}\right)^2 - 6 = 0$$

$$\frac{1}{5}(4X^2+Y^2-4XY) + \frac{4}{5}(2X^2+4XY-XY-2Y^2) - \frac{2}{5}(X^2+4Y^2+4XY) - 6 = 0$$

$$10X^2 - 15Y^2 = 30$$

$$\frac{X^2}{3} - \frac{Y^2}{3} = 1 \quad ..... (4) \quad \text{which represents hyperbola}$$

$$a^2 = 3, \quad b^2 = 2$$

For center Put $X = 0$, $Y = 0$

$$\Rightarrow \frac{2x+y}{\sqrt{5}} = 0 \Rightarrow 2x + y = 0 \quad \& \quad Y = 0$$

$$\frac{2x-y}{\sqrt{5}} = 0 \Rightarrow -x + 2y = 0$$

Solving $2x + y = 0$ & $-x + 2y = 0$

we get $x = 0$, $y = 0$

Thus center (0, 0)

For transverse axis put $Y = 0$ we get

$$x - 2y = 0$$

Conjugate axis put $X = 0$, we get $2x + y = 0$

Vertices

$$(X, Y) = (\pm\sqrt{3}, 0)$$

$$\frac{2x+y}{\sqrt{5}} = \pm\sqrt{15} \quad \text{(i)} \qquad \frac{2y-x}{\sqrt{5}} = 0 \Rightarrow x = 2y \quad .... \text{(ii)}$$

By solution of (i) & (ii) we get

$$\left(2\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right), \quad \left(-2\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right)$$

Foci $(X, Y) = (\pm\sqrt{5}, 0)$

$X = \pm\sqrt{5} \qquad Y = 0$

$2x + y = \pm 5 \quad , \quad -x + 2y = 0$

After solution of above equations

$(2, 1)$ & $(-2, -1)$

(ix) $\mathbf{x^2 - 4xy - 2y^2 + 10x + 4y = 0}$ **(1)**

Solution:

$a = 1 \quad , \quad b = -2 \qquad 2h = -4$

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{3}$$

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{-4}{3}$$ After cross multiplication we get

$2\tan^2\theta - 3\tan\theta - 2 = 0$

$2\tan^2\theta - 4\tan\theta + \tan\theta - 2 = 0$

$2\tan\theta(\tan\theta - 2) + 1(\tan\theta - 2) = 0$

$\tan\theta - 2 = 0 \qquad 2\tan\theta + 1 = 0$

$\tan\theta = 0 \qquad \cot\theta = \frac{1}{2}$

$\sec\theta = \sqrt{5} \qquad \cos\theta = \frac{1}{\sqrt{5}} \qquad \sin\theta = \frac{2}{\sqrt{5}}$

Now equations of transformation are

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = \frac{X-2Y}{\sqrt{5}} \\ x &= X\sin\theta + Y\cos\theta = \frac{2X-Y}{\sqrt{5}} \end{aligned}\right] \quad (2) \qquad \text{Put in (1)}$$

$$\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X-Y}{\sqrt{5}}\right) - 2\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 10\left(\frac{X-2Y}{\sqrt{5}}\right) + 4\left(\frac{2X-Y}{\sqrt{5}}\right) = 0$$

$$\Rightarrow \quad -15X^2 + 10Y^2 + 18\sqrt{5}X - 16\sqrt{5}Y = 0$$

$$\Rightarrow \quad 10\left(Y^2 - \frac{16\sqrt{5}}{10}Y\right) - 15\left(X^2 - \frac{18\sqrt{5}}{15}X\right) = 0$$

$$10\left(Y^2 - \frac{8\sqrt{5}}{5}Y\right) - 15\left(X^2 - \frac{6\sqrt{5}}{5}X\right) = 0$$

$$10\left[Y^2 - \frac{8\sqrt{5}}{5}Y + \left(\frac{4}{\sqrt{5}}\right)^2\right] - 15\left[X^2 - \frac{6\sqrt{5}}{5}X + \left(\frac{3}{\sqrt{5}}\right)^2\right] = 10\left(\frac{16}{5}\right) - 15\left(\frac{9}{5}\right)$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = 5$$

$$\frac{\left(Y - \frac{4}{\sqrt{5}}\right)^2}{\frac{1}{2}} - \frac{\left(X - \frac{3}{\sqrt{5}}\right)^2}{\frac{1}{3}} = 1 \quad \text{..... (3)} \quad \text{Hyperbola}$$

$$a^2 = \frac{1}{2} \quad \& \quad b^2 = \frac{1}{3}$$

$$e^2 = \frac{a^2 + b^2}{a^2} = \frac{5}{3} \quad \Rightarrow \quad e = \sqrt{\frac{5}{3}}$$

Center put

$$Y - \frac{4}{5} = 0 \Rightarrow Y = \frac{4}{\sqrt{5}}, \quad X - \frac{3}{\sqrt{5}} = 0 \Rightarrow X = \frac{3}{\sqrt{5}}$$

$$X - 2Y = \sqrt{5}\,x \quad , \quad X - 2Y = \sqrt{5}\,x$$

$$2X + Y = \sqrt{5}\,y \quad , \quad 2X + Y = \sqrt{5}\,y$$

Multiplying 2nd equation by 2

$$X - 2Y = \sqrt{5}\,x$$
$$4X + 2Y = 2\sqrt{5}\,y$$
$$5X = \sqrt{5}\,(x + 2y)$$
$$X = \frac{1}{\sqrt{5}}(x + 2y)$$
$$\frac{3}{\sqrt{5}} = \frac{1}{\sqrt{5}}(x + 2y)$$
$$x + 2y = 3$$

Multiplying equation 1st by 2

$$2X - 4Y = 2\sqrt{5}\,x$$
$$-2X \pm Y = -\sqrt{5}\,y$$
$$-5y = \sqrt{5}\,(2x - y)$$
$$Y = -\frac{1}{\sqrt{5}}(2x - y)$$
$$-\frac{1}{\sqrt{5}}(2x - y) = \frac{4}{\sqrt{5}}$$
$$-2x + y = 4$$

$$-2x + y = 4$$

After solution we get

Center $(-1, 2)$

Foci $= (0, \pm ae)$

$$\left(X - \frac{3}{\sqrt{5}} \quad , \quad Y - \frac{4}{\sqrt{5}}\right) = \left(0 \quad , \quad \pm \frac{\sqrt{5}}{\sqrt{6}}\right)$$

$$X - \frac{3}{\sqrt{5}} = 0 \qquad\qquad Y - \frac{4}{\sqrt{5}} = \pm \frac{\sqrt{5}}{\sqrt{6}}$$

$$X = \frac{3}{\sqrt{5}} \qquad\qquad Y = \frac{4}{\sqrt{5}} \pm \frac{\sqrt{5}}{\sqrt{6}}$$

$$\frac{1}{\sqrt{5}}(x + 2y) = \frac{3}{\sqrt{5}} \qquad\qquad \frac{-1}{\sqrt{5}}(2x - 2y) = \frac{4}{\sqrt{5}} + \frac{\sqrt{5}}{\sqrt{6}}$$

$$x + 2y = 3 \qquad (1)' \qquad\qquad -2x + y = 4 \pm \frac{5}{\sqrt{6}}$$

After simplification foci are

$$\left(-1 - \frac{2}{\sqrt{6}}, 2 + \frac{1}{\sqrt{6}}\right) \quad \& \quad \left(-1 + \frac{2}{\sqrt{6}}, 2 - \frac{1}{\sqrt{6}}\right)$$

Q.2: Show that

$10xy + 8x - 15y - 12 = 0$ represents a pair of straight lines and find an equation of each line.

Solution:

$10xy + 8x - 15y - 12 = 0$ ..... (1)

Compare it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 0, \quad h = 5, \quad b = 0, \quad g = 4, \quad f = \frac{-15}{2} \quad C = -12$

Now consider determinant $= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & \frac{-15}{2} \\ 4 & \frac{-15}{2} & -12 \end{vmatrix}$

$$= 0\begin{vmatrix} 0 & \frac{-15}{2} \\ \frac{-15}{2} & -12 \end{vmatrix} - 5\begin{vmatrix} 5 & \frac{-15}{2} \\ 4 & -12 \end{vmatrix} + 4\begin{vmatrix} 5 & 0 \\ 4 & \frac{-15}{2} \end{vmatrix}$$

$$= 0 - 5(-60 + 30) + 4\left(\frac{-75}{2}\right) \Rightarrow 150 - 150 = 0$$

This shows that equation (1) represents a pair of straight line, now we'll solve (1) as

$$10\,xy + 8x - 15y - 12 = 0$$
$$2x\,(5y + 4) - 3\,(5y + 4) = 0$$
$$(5y + 4)\,(2x - 3) = 0$$
$$5y + 4 = 0 \;,\quad 2x - 3 = 0$$

(ii) $\mathbf{6x^2 + xy - y^2 - 21x - 8y + 9 = 0}$ **(1)**

Solution:

Comparing by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$a = 6,\quad h = \frac{1}{2},\quad b = -1\;,\quad g = \frac{-21}{2}\;,\quad f = -4,\quad c = 9$$

Now consider determinant $= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$= \begin{vmatrix} 6 & \frac{1}{2} & \frac{-21}{2} \\ \frac{1}{2} & -1 & -4 \\ \frac{-21}{2} & -4 & 9 \end{vmatrix}$$

$$= 6\begin{vmatrix} -1 & -4 \\ -4 & 9 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \frac{1}{2} & -4 \\ \frac{-21}{2} & 9 \end{vmatrix} - \frac{21}{2}\begin{vmatrix} \frac{1}{2} & -1 \\ \frac{-21}{2} & -4 \end{vmatrix}$$

$$= 6(-9 - 16) - \frac{1}{2}\left(\frac{9}{2} - \frac{84}{2}\right) - \frac{21}{2}\left(\frac{-4}{2} - \frac{21}{2}\right)$$

$$= 6(-25) - \frac{1}{2}\left(\frac{-75}{2}\right) - \frac{21}{2}\left(\frac{-25}{2}\right)$$

$$= -150 + \frac{75}{4} + \frac{525}{4}$$

$$= -150 + 150 = 0$$

This shows that given equation (1) represents a pair of straight lines. Now re-arranging (1) we have

$$6x^2 + (y - 21)x - (y^2 + 8y - 9) = 0$$

By Quadratic formula

$$x = \frac{-(y-21) \pm \sqrt{(y-21)^2 + 4(6)(y^2+8y-9)}}{2(6)}$$

$$x = \frac{-(y-21) \pm \sqrt{y^2 + 441 - 42y + 24y^2 + 192y - 216}}{12}$$

$$x = \frac{-y+21 \pm \sqrt{25y^2 + 150y + 225}}{12}$$

$$x = \frac{-y+21 \pm \sqrt{(5y+15)^2}}{12} = \frac{-y+21 \pm (5y+15)}{12}$$

$$x = \frac{-y+21+5y+15}{12}, \quad \frac{-y+21-5y-15}{12}$$

$$x = \frac{4y+36}{12} \qquad x = \frac{-6y+6}{12}$$

$$x = \frac{4(y+9)}{12} \qquad x = \frac{6(-y+1)}{12}$$

$$x = \frac{y+9}{3} \qquad x = \frac{-y+1}{2}$$

$3x - y - 9 = 0$, $2x + y - 1 = 0$ Ans.

Q.3: Find an equation of tangent to each of the given conics at indicated point.

(i) $3x^2 - 7y^2 + 2x - y - 48 = 0$ at (4, 1)

Solution:

$3x^2 - 7y^2 + 2x - y - 48 = 0$ at (4, 1)

Diff. w.r.t 'x'

$$6x - 14y\frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$

$$(-14y - 1)\frac{dy}{dx} = -6x - 2$$

$$\frac{dy}{dx} = \frac{-6x-2}{-14y-1} = \frac{6x+2}{14y+1}$$

$$m = \frac{dy}{dx}\Big|_{(4,1)} = \frac{6(4)+2}{14(1)+1} = \frac{26}{15}$$

Equation of tangent at the point (4, 1) is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{26}{15}(x - 4)$$

$15y - 15 = 26x - 104$

=> $\boxed{26x - 15y - 89 = 0}$

(ii) $\mathbf{x^2 + 5xy - 4y^2 + 4 = 0}$ **at** $\mathbf{y = -1}$

Solution:

$x^2 + 5xy - 4y^2 + 4 = 0$

Put $y = -1$

$x^2 - 5x - 4 + 4 = 0$

$x(x - 5) = 0$

$x = 0 \quad , \quad x = 5$

We have two points $(0, -1)$ & $(5, -1)$

Now diff (1) w.r.t 'x'

$$2x + 5\left[y + \frac{dy}{dx}x\right] - 8y\frac{dy}{dx} = 0$$

$$2x + 5y + 5x\frac{dy}{dx} - 8y\frac{dy}{dx} = 0$$

$$(5x - 8y)\frac{dy}{dx} = -2x - 5y$$

$$\frac{dy}{dx} = \frac{-(2x + 5y)}{-(8y - 5x)}$$

$$m = \frac{dy}{dx}\Big|_{(0,-1)} = \frac{2(0) + 5(-1)}{8(-1) - 5(0)} = \frac{-5}{-8} = \frac{5}{8}$$

Equation of tangent at point $(0, -1)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 1 = \frac{5}{8}(x - 0)$$

$$8y + 8 = 5x \quad \Rightarrow \quad \boxed{5x - 8y - 8 = 0} \quad \text{Ans}$$

Next

$$m = \frac{dy}{dx}\Big|_{(5,-1)} = \frac{2(5) + 5(-1)}{8(-1) - 5(5)} = \frac{10 - 5}{-8 - 25} = \frac{5}{-33}$$

Equation of tangent at $(5, -1)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 1 = \frac{-5}{33}(x - 5)$$

$$33y + 33 = -5x + 25$$

$$\boxed{5x + 33y = -8} \quad \text{Ans}$$

(iii) $\mathbf{x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0}$ **at** $\mathbf{x = 3}$

Solution:

$$x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0 \quad ..... \quad (1)$$

Put $x = 3$

$$9 + 12y - 3y^2 - 15 - 9y + 6 = 0$$

$$-3y^2 + 3y = 0$$

$$-3y(y - 1) = 0$$

=> $y = 0$, $y = 1$

Required points are (3, 0) & (3, 1)

Diff. (1) w.r.t 'x'

$$2x + 4[y + x\frac{dy}{dx}] - 3(2y\frac{dy}{dx}) - 5 - 9\frac{dy}{dx} = 0$$

$$2x + 4y + 4x\frac{dy}{dx} - 6y\frac{dy}{dx} - 5 - 9\frac{dy}{dx} = 0$$

$$(4x - 6y - 9)\frac{dy}{dx} = -2x - 4y + 5$$

$$\frac{dy}{dx} = \frac{-2x - 4y + 5}{4x - 6y - 9}$$

$$m = \frac{dy}{dx}\Big|_{(3,0)} = \frac{-2(3) - 4(0) + 5}{4(3) - 6(0) - 9}$$

$$m = \frac{-6 + 5}{12 - 9} = \frac{-1}{3}$$

Equation of tangent at (3, 0)

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{-1}{3}(x - 3)$$

$$3y = -x + 3 \quad => \quad \boxed{x + 3y - 3 = 0} \quad \text{Ans.}$$

Next, $m = \frac{dy}{dx}\Big|_{(3,1)} = \frac{-2(3) - 4(1) + 5}{4(3) - 6(1) - 9} = \frac{-6 - 4 + 5}{12 - 6 - 9} = \frac{5}{3}$

Equation of tangent at (3, 1)

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{5}{3}(x - 3)$$

$$3y - 3 = 5x - 15$$

$$\boxed{5x - 3y - 12 = 0}$$